GENERALIZED SPLINES FOR RADON TRANSFORM ON COMPACT LIE GROUPS WITH APPLICATIONS TO CRYSTALLOGRAPHY

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Abstract. The Radon transform $\mathcal{R}f$ of functions f on $SO(3)$ has recently been applied extensively in texture analysis, i.e. the analysis of preferred crystallographic orientation. In practice one has to determine the orientation probability density function $f \in L_2(SO(3))$ from $\mathcal{R}f \in L_2(S^2 \times S^2)$ which is known only on a discrete set of points. Since one has only partial information about $\mathcal{R}f$ the inversion of the Radon transform becomes an ill-posed inverse problem. Motivated by this problem we define a new notion of the Radon transform $\mathcal{R}f$ of functions f on general compact Lie groups and introduce two approximate inversion algorithms which utilize our previously developed generalized variational splines on manifolds. Our new algorithms fit very well to the application of Radon transform on $SO(3)$ to texture analysis.

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1. Introduction

The objective of the present paper is to introduce Radon transform on compact Lie groups and to show how spline interpolation can be used for approximate inversion of such transform on general compact Lie groups and in particular on the group of rotations $SO(3)$.

The Radon $\mathscr{R}$ transform on a compact $\mathcal{G}$ Lie group associated with a closed subgroup $\mathscr{H}$ assigns to a smooth function f integrals over submanifolds of $\mathcal{G}$ which have the form $x\mathscr{H}y^{-1}$ where $x, y \in \mathcal{G}$. An important fact is that the Radon transform $\mathscr{R}f$ is a function on $\mathcal{G}/\mathscr{H} \times \mathcal{G}/\mathscr{H}$ and not merely on $\mathcal{G} \times \mathcal{G}$ as one could expect (see below for details). The typical problem is to reconstruct f knowing $\mathscr{R}f$. In practice one has only integrals over manifolds $\{\mathcal{M}_\nu\}_1^N$ from a **finite** subfamily. In this situation inversion becomes an ill posed problem. Our objective is to consider appropriate regularization of this ill posed problem and to develop a method for approximate inversion of the Radon transform.

In our approach to approximate inversion of the Radon transform (section 5) we consider inversion as an interpolation problem. Namely, given a set of integrals

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of an f over a finite family $\mathcal{M} = \{\mathcal{M}_\nu\}_1^N$ of submanifolds, we find a "smoothest" function which has the same set of integrals as f over submanifolds from the family $\mathcal{M}$.

Our work is inspired by recent applications of the Radon transform on $SO(3)$ to texture analysis, i.e. the analysis of preferred crystallographic orientation.

The orientation probability density function (ODF) representing the probability law of random orientations of crystal grains by volume is a major issue. In x-ray or neutron diffraction experiments spherical intensity distributions are measured which can be interpreted in terms of spherical probability distributions of distinguished crystallographic axes. A first mathematical description of the problem was given in [7].

One equips a crystal with an inner orthogonal coordinate system $\{e_1, e_2, e_3\}$. Additionally one distinguish an outer orthogonal coordinate system $\{u_1, u_2, u_3\}$ related to the specimen. The orientation of a crystal in the specimen is defined by the unique rotation $\gamma \in SO(3)$ which maps the inner coordinate system to the outer one, i.e. $ge_i = u_i$ for $i = 1, 2, 3$. Note that in this model we neglect the spherical symmetries of the crystal.

The function of interest is the *orientation density function (ODF)* $f \in L^2(SO(3))$ that is the probability measure on $SO(3)$. Hence the function value $f(g)$ gives the portion of crystals in the specimen with orientation g.

The practical measurement sends a beam through the specimen coming from the direction $h \in S^2$ and measures the intensity, emitted from the specimen in the direction $r \in S^2$. One can interpret the result as the integral over all orientations $g \in SO(3)$ with $g \cdot h = r$; $h, r \in S^2$. The set $C_{h,r} = \{g \in SO(3) : g \cdot h = r;\ h, r \in S^2\}$ of those orientations is called a great circle in $SO(3)$.

Since $SO(2)$ is the stabilizer of $\xi_0 \in S^2$, where ξ_0 is the north pole one has

$$C_{h,r} = h'\, SO(2)\, r'^{-1} := \{h' g r'^{-1},\ g \in SO(2)\} \qquad h', r' \in SO(3), \tag{1}$$

where $h', r' \in SO(3)$ satisfy $h' \cdot \xi_0 = h$ and $r' \cdot \xi_0 = r$. Here (x, y) is a fixed point in $S^2 \times S^2$.

Definition 1.1. *The Radon transform of a continuous complex-valued function f on $SO(3)$ is a function on $S^2 \times S^2$ which is defined by the formula*

$$\mathscr{R} f(x, y) = \int_{C_{x,y}} f(g)\, \mathrm{d}x. \tag{2}$$

This transform $\mathscr{R}$ can be extended to all functions in $L^2(SO(3))$.

From the very definition $\mathscr{R} f$ is a function on $S^2 \times S^2$. Moreover, it is shown that $\mathscr{R} f$ is in the kernel of the Darboux-type differential operator (also called ultra-hyperbolic operator). In other words, one has

$$\Delta_x \mathscr{R} f(x, y) = \Delta_y \mathscr{R} f(x, y), \quad (x, y) \in S^2 \times S^2.$$

It is interesting to note that this condition is similar to the classical condition of F. John [15] for X-ray transform in $\mathbb{R}^3$.

The Darboux-type equation shows that $\mathscr{R} f$ belongs to the span of the tensorial product of spherical harmonics $\mathcal{Y}_k^i \mathcal{Y}_k^j$ which is obviously a subspace of $L_2(S^2 \times S^2)$ whose basis is $\mathcal{Y}_k^i \mathcal{Y}_l^j$.

Because S^3 is a double covering of $SO(3)$ there is also a close connection to the spherical Radon transform of even functions considered by S. Helgason ([18], [19])

as a special case of Radon tranforms on homogeneous spaces. A comparison of $\mathscr{R}$ to the spherical Radon transform of even functions is given in [4].

The numerical aspect of solving the crystallographic problem as well as the inversion of $\mathscr{R}$ is treated in [6] and [20].

To tackle the real-world problem of sharp textures, i.e. the pole figures consists mainly of a few delta peaks, wavelets on the sphere S^3 and the group $SO(3)$ and their behavior under $\mathscr{R}$ had been studied in [3] and [10]. In [11] and [10] an grouptheoretical approach is used to generalize the results for wavelets on groups and homogeneous spaces which leads to a new class of wavelets, called diffusive wavelets.

Another approach to solve the inversion problem of $\mathscr{R}$ is done in [8], where Gabor frames on the $Spin(3)$ had been used to solve the problem numerically.

In section 2 of the paper we use a group theoretic and representation theoretic approach to describe the Radon transform on a general compact Lie group $\mathcal{G}$. At first hand $(\mathscr{R}f)(x,y)$ seems to be defined over $\mathcal{G}\times\mathcal{G}$ while deeper inverstigations reveal that $\mathscr{R}f$ is invariant under right shifts of x as well as y and hence $\mathscr{R}$ is rather defined over $\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H}$. For the practical application $\mathcal{G}=SO(3)$ and $\mathscr{H}=SO(2)$ and thus $\mathcal{G}/\mathscr{H}=SO(3)/SO(2)=S^2$. In this particular situation a number of explicit inversion formulas is known. For more details we refer to [2], [4], [5], [7], [21], [22], [24]- [29].

In section 4 of the paper we discuss generalized variational splines on compact Riemannian manifolds (see [30], [34], [33]) and apply this concept to the Radon transform.

In [30]-[34] a theory of generalized (=average) interpolating variational splines was developed in compact and non-compact manifolds of bounded geometry. We use term generalized splines to stress that we interpolate functions on manifolds by using values of their integrals over submanifolds of a given manifold. It includes the classical way of interpolating functions is by using their values on discrete sets of points. Variational splines on manifolds were used in our papers for reconstruction of Paley-Wiener functions on manifolds. In [34] the theory of generalized variational splines was applied to the spherical Radon transform of even functions (Funk transform) and to the hemispherical transform of odd functions on n-dimensional spheres.

The development in the present paper is slightly different from our approach in previous papers in the sense that here we introduce splines using general elliptic second order self-adjoint operators and not only the Laplace-Beltrami operator. We prove existence and uniqueness of interpolating variational splines associated with a general elliptic second order self-adjoint operator on a compact Riemannian manifold. We remind (see Theorem 4.6) that every generalized spline is a linear combinations of what we call generalized fundamental solutions (= generalized Green functions). Using these results "explicit" formulas of variational splines in terms of eigenfunctions of the corresponding operator are given.

In section 3 we develop our first method for approximate inversion of Radon transform on compact Lie groups. Note that in a similar situation for Radon a hemispherical transforms on unit spheres this idea was introduced in [34]. The idea is that one can always have an approximate inversion of "any kind" of Radon transform by treating inversion as a generalized interpolation problem. Thus, we

invert group Radon transform by "interpolating" a function of the group using its integrals over a finite family of appropriate submanifolds.

In the last section we review our fundamental inequalities (Lemma 6.2, Theorem 6.3) for functions with many zeroes (see [30], [34], [33]) and our Approximation Theorem for interpolating variational splines on general compact manifolds ([34], [33]). The rate of approximation in this theorem is expressed in terms of Sobolev norms. It is worth to note that using standard methods of interpolation theory and the fact that Besov spaces on manifolds are interpolation spaces (real interpolation method) between two Sobolev spaces the corresponding inequalities can be extended to Besov norms.

We use these results to describe our second method of approximate inversion of the Radon transform on $SO(3)$. The idea of this method is to interpolate not the function but its Radon transform on the manifold $S^2 \times S^2$ and then "to return" to the original manifold $SO(3)$. This method allows to obtain explicit estimates of degree of approximation in Theorem 6.5. Moreover, in the same section we establish a Sampling Theorem for Radon transform on $SO(3)$ (Theorem 6.6) which says that one can have a complete reconstruction of ω-bandlimited (on $SO(3)$) function f by using only the values of its integrals over sufficiently dense specific family of submanifolds of $SO(3)$.

It should be stressed that the variational spline problem fits in some sense optimal to the practical question of determining the ODF f from measurements of the Radon transformed f. On the one hand we are interested in regions where the values of f are large but if the curvature of f is small in those regions it would be more useful to increase the measurements around the maximum of f and those points where the curvature is large. Hence the right criteria to increase the density of measurements around points where $(1-\Delta)f$ is large. The density of measurements should be high at those points where the interpolation is highly nonlinear.

It should be noted that a very different and more constructive approach to splines on two-dimensional surfaces was developed in [1], [9], [12]-[14], [23].

An approach to splines on compact manifolds which is similar to our approach in [30]-[34] (in the sense it is based on a "zeroes lemma" and Green's function representations) was recently developed in [16], [17] for the classical way of interpolation on discrete sets of points.

2. Fourier analysis and Radon transform on compact Lie groups

2.1. **Fourier Analysis on compact groups.** One of the most important theorems of functional analysis is the spectral theorem for compact self-adjoint operators on a Hilbert space. Which states that if A is a compact self-adjoint operator on a Hilbert space V, then there is an orthonormal basis of V consisting of eigenvectors of A and each eigenvalue is real. The analog of this theorem is the Peter-Weyl theorem. We want to recollect some basic notations and properties.

Let $\mathcal{G}$ be a compact Lie group. A unitary representation of $\mathcal{G}$ is a continuous group homomorphism π: $\mathcal{G} \to U(d_\pi)$ of $\mathcal{G}$ into the goup of unitary matrices of a certain dimension d_π which will be explained later in the Peter-Weyl theorem. Such a representation is irreducible if $\pi(g)M = M\pi(g)$ for all $g \in \mathcal{G}$ and some $M \in \mathbb{C}^{d_\pi \times d_\pi}$ implies $M = cI$ is a multiple of the identity. Equivalently, $\mathbb{C}^{d_\pi}$ does not have non-trivial π-invariant subspaces $V \subset \mathbb{C}^{d_\pi}$ with $\pi(g)V \subset V$ for all $g \in \mathcal{G}$.

Two representations π_1 and π_2 are equivalent, if there exists an invertible matrix M such that $\pi_1(g)M = M\pi_2(g)$ for all $g \in \mathcal{G}$.

Let $\hat{\mathcal{G}}$ denote the set of all equivalence classes of irreducible representations. Then this set parametrerizes an orthogonal decomposition of $L^2(\mathcal{G})$.

Theorem 2.1 (Peter-Weyl, [35])**.** *Let $\mathcal{G}$ be a compact Lie group. Then the following statements are true.*

a: *Denote $\mathcal{H}_\pi = \{g \mapsto \mathrm{trace}(\pi(g)M) : M \in \mathbb{C}^{d_\pi \times d_\pi}\}$. Then the Hilbert space $L^2(\mathcal{G})$ decomposes into the orthogonal direct sum*

$$L^2(\mathcal{G}) = \bigoplus_{\pi \in \hat{\mathcal{G}}} \mathcal{H}_\pi \tag{3}$$

b: *For each irreducible representation $\pi \in \hat{\mathcal{G}}$ the orthogonal projection $L^2(\mathcal{G}) \to \mathcal{H}_\pi$ is given by*

$$f \mapsto d_\pi \int_{\mathcal{G}} f(h)\chi_\pi(h^{-1}g)\,dh = d_\pi\, f * \chi_\pi, \tag{4}$$

in terms of the character $\chi_\pi(g) = \mathrm{trace}(\pi(g))$ of the representation and dh is the normalized Haar measure.

We will denote the matrix M in the equation $f * \chi_\pi = \mathrm{trace}(\pi(g)M)$ as Fourier coefficient $\hat{f}(\pi)$ of f at the irreducible representation π. The Fourier coefficient can be calculated as

$$\hat{f}(\pi) = \int_{\mathcal{G}} f(g)\pi^*(g)\,dg.$$

The inversion formula (the Fourier expansion) is then given by

$$f(g) = \sum_{\pi \in \hat{G}} d_\pi\, \mathrm{trace}(\pi(g)\hat{f}(\pi)).$$

If we denote by $||M||_{HS}^2 = \mathrm{trace}(M^*M)$ the Frobenius or Hilbert-Schmidt norm of a matrix M, then the following Parseval identity is true.

Corollary 2.1 (Parseval identity)**.** *Let $f \in L^2(\mathcal{G})$. Then the matrix-valued Fourier coefficients $\hat{f} \in \mathbb{C}^{d_\pi \times d_\pi}$ satisfy*

$$||f||^2 = \sum_{\pi \in \hat{\mathcal{G}}} d_\pi\, ||f(\pi)||_{HS}^2. \tag{5}$$

On the group $\mathcal{G}$ one defines the convolution of two integrable functions $f, r \in L^1(\mathcal{G})$ as

$$f * r(g) = \int_{\mathcal{G}} f(h)r(h^{-1}g)\,dh.$$

Since $f * r \in L^1(\mathcal{G})$, the Fourier coefficients are well-defined and they satisfy

Corollary 2.2 (Convolution theorem on $\mathcal{G}$)**.** *Let $f, r \in L^1(\mathcal{G})$ then $f * r \in L^1(\mathcal{G})$ and*

$$\widehat{f * r}(\pi) = \hat{f}(\pi)\hat{r}(\pi).$$

The group structure gives rise to left and right translations $T_g f \mapsto f(g^{-1}\cdot)$ and $T^g f \mapsto f(\cdot g)$ of functions on the group. A simple computation shows

$$\widehat{T_g f}(\pi) = \hat{f}(\pi)\pi^*(g) \quad \text{and} \quad \widehat{T^g f}(\pi) = \pi(g)\hat{f}(\pi).$$

They are direct consequences of the definition of the Fourier transform.

The Laplace-Beltrami operator $\Delta_{\mathcal{G}}$ on the group $\mathcal{G}$ is bi-invariant, i.e. commutes with all T_g and T^g. Therefore, all its eigenspaces are bi-invariant subspaces of $L^2(\mathcal{G})$. As $\mathcal{H}_\pi$ are minimal bi-invariant subspaces, each of them has to be eigenspace of $\Delta_{\mathcal{G}}$ with corresponding eigenvalue $-\lambda_\pi^2$. Hence, we obtain

$$\Delta_{\mathcal{G}} f = -\sum_{\pi\in\hat{G}} d_\pi\, \lambda_\pi^2 \,\mathrm{trace}(\pi(g)\hat{f}(\pi)).$$

Motivated by situation which was described we introduce a new type of Radon transform which is an abstract version of the crystallographic Radon transform.

2.2. **Radon transform on compact groups.** Now, we are able to define the Radon transform.

Definition 2.2. *Let $\mathscr{H}$ be a closed subgroup of the compact Lie group $\mathcal{G}$. The Radon transform of a continuous function $f \in C(\mathcal{G})$ is defined by*

$$\mathscr{R}f(x,y) = \int_{\mathscr{H}} f(xhy^{-1})\,\mathrm{d}h \qquad x,y\in\mathcal{G}, \tag{6}$$

where dh here is the normalized Haar measure on $\mathscr{H}$.

Next, we explain that $\mathscr{R}$ maps $\mathcal{G}$ into $\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H}$. For that we use the averaging method.

For the following discussion we mention, that functions on $\mathcal{G}/\mathscr{H}$ can be regarded as functions on $\mathcal{G}$, which are constant over right co-sets of the form $g\mathscr{H} = \{gh,\, h\in\mathscr{H}\}$ for all $g\in\mathcal{G}$. The projection of a function on $\mathcal{G}$ onto functions on $\mathcal{G}/\mathscr{H}$ corresponds to an averaging method over $g\mathscr{H}$:

$$\mathbb{P}_{\mathscr{H}} f(g) = \int_{\mathscr{H}} f(gh)\,\mathrm{d}h. \tag{7}$$

It can be shown, that $\mathbb{P}_{\mathscr{H}}$ in Fourier domain acts by multiplying the Fourier coefficients $\hat{f}(\pi)$ by

$$\pi_{\mathscr{H}} = \int_{\mathscr{H}} \pi(h)\,\mathrm{d}h \tag{8}$$

from the right. Further $\pi_{\mathscr{H}}$ is a projection and without loss of generality $\pi_{\mathscr{H}} = diag(1,...,1,0,..,0)$, where the number of 1's corresponds to the number of $\mathscr{H}$ invariant vectors in the representation Hilbert space of π. For details on the projection see [36]. Next we discuss the Range of $\mathscr{R}$. Since x,y in (6) are elements of $\mathcal{G}$ at a first look it seems that the Radon transform is defined over $\mathcal{G}\times\mathcal{G}$. While a deeper investigation reveals that $\mathscr{R}f(x,y)$ is invariant under right shifts of x as well as under right shifts of y, hence $\mathscr{R}$ is rather defined over $\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H}$.

Lemma 2.3. *The Radon transform $\mathscr{R}$ maps functions over $\mathcal{G}$ to functions over $\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H}$.*

Proof. We look at $\mathscr{R}$ on the Fourier domain. Let first $y\in\mathcal{G}$ be fixed and regard $\mathscr{R}f(\cdot,y)$ to be a function on $\mathcal{G}$ in the first argument, then

$$\widehat{\mathscr{R}f(\cdot,y)}(\pi) = \pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\pi\in\widehat{\mathcal{G}}, \tag{9}$$

Hence the the function $\mathscr{R}f(\cdot,y)$ is invariant under the projection $\mathbb{P}_{\mathscr{H}}$, since the Fourier coefficients are invariant under the left multiplication by $\pi_{\mathscr{H}}$:

$$\pi_{\mathscr{H}}\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi) = \pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi).$$

Consequently,

$$\mathscr{R}f(x\cdot h, y) = \mathscr{R}f(x,y) \;\; h \in \mathscr{H}. \tag{10}$$

A look at the Radon transform as function in the second argument y, while the first one $x = x_0$ is fixed, we find

$$\mathbb{P}_{\mathscr{H}}\mathscr{R}f(x_0,y) = \int_{\mathscr{H}} \mathscr{R}f(x_0,yh)\,dxh =$$
$$\int_{\mathscr{H}} \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi(\widehat{f}(\pi)\pi(x_0))\pi_{\mathscr{H}}\pi(h^{-1}y^{-1})\,dxh$$
$$= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi(\widehat{f}(\pi)\pi(x_0))\pi_{\mathscr{H}}\pi^*(y) = \mathscr{R}f(x_0,y),$$

Hence $\mathscr{R}f(x,y)$ is constant over fibers of the form $y\mathscr{H}$ also in the second argument and

$$\widehat{\mathscr{R}f(x,\cdot)}(\pi) = \pi_{\mathscr{H}}\overline{\pi^*(x)\widehat{f}(\pi)^*}, \tag{11}$$

where the complex conjugate of the matrices is taken componentwise. Consequently $\mathscr{R}$ maps functions over $\mathcal{G}$ to functions over $\mathcal{G}/\mathscr{H} \times \mathcal{G}/\mathscr{H}$. □

Below we discuss the Radon transform of the space $L^2(\mathcal{G})$, its range and its inversion.

Lemma 2.4. *Let $\mathscr{H}$ be the subgroup of $\mathcal{G}$, determining the Radon transform on $\mathcal{G}$ and let $\widehat{\mathcal{G}}_1 \subset \widehat{\mathcal{G}}$ be the set of irreducible representations with respect to $\mathscr{H}$. Then for $f \in C^\infty(\mathcal{G})$ it is*

$$\|\mathscr{R}f\|^2_{L^2(\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H})} = \sum_{\pi\in\widehat{\mathcal{G}}_1} rank(\pi_{\mathscr{H}})\|\widehat{f}(\pi)\|^2_{HS}. \tag{12}$$

Proof. For the proof we expand $\mathscr{R}f(x,y)$ for fixed y as function in x over $\mathcal{G}$ (or better $\mathcal{G}/\mathscr{H}$) and apply Parseval's identity (5), with (9) we have

$$\begin{aligned}\|\mathscr{R}f\|^2_{L^2(\mathcal{G}/\mathscr{H}\times\mathcal{G}/\mathscr{H})} &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \int_{\mathcal{G}} \|\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\|^2_{HS}\,\mathrm{d}y\\ &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \int_{\mathcal{G}} \mathrm{trace}\left(\widehat{f}^*(\pi)\pi(y)\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\right)\,\mathrm{d}y\\ &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi\,\mathrm{trace}\left(\widehat{f}^*(\pi)\int_{\mathcal{G}} \pi(y)\pi_{\mathscr{H}}\pi^*(y)\,\mathrm{d}y\,\widehat{f}(\pi)\right)\\ &= \sum_{\pi\in\widehat{\mathcal{G}}_1} \mathrm{rank}(\pi_{\mathscr{H}})\,\mathrm{trace}(\widehat{f}^*\widehat{f}) = \sum_{\pi\in\widehat{\mathcal{G}}_1} \mathrm{rank}(\pi_{\mathscr{H}})\|\widehat{f}(\pi)\|^2_{HS}.\end{aligned}$$

Where we made use of

$$\begin{aligned}\int_{\mathcal{G}} \widehat{\pi}(y)\pi_{\mathscr{H}}\pi^*(y)\,\mathrm{d}y &= \left(\sum_{k=1}^{\mathrm{rank}_{\pi_{\mathscr{H}}}} \int_{\mathcal{G}} \pi_{ik}(y)\overline{\pi_{kj}(y)}\,\mathrm{d}y\right)_{i,j=1}^{d_\pi}\\ &= \frac{\mathrm{rank}(\pi_{\mathscr{H}})}{d_\pi} Id.\end{aligned} \tag{13}$$

□

3. Harmonic analysis and Radon transform on $SO(3)$

3.1. **Special functions of $SO(3)$.** For the Hilbert space $L^2(S^2)$ we use the orthonormal system of spherical harmonics in the $\{\mathcal{Y}_k^i,\ i = 1, ..., 2k+1, k \in \mathbb{N}_0\}$.

In case of $SO(3)$ the situation is very comfortable, since every irreducible representation is unitary equivalent to a irreducible component of the quasi regular representation in $L^2(S^2)$, given by

$$T(g) : f(\xi) \mapsto f(g^{-1} \cdot x), \tag{14}$$

where $\cdot$ denotes the canonical action of $SO(3)$ on S^2. The irreducible invariant components of $L^2(S^2)$ under T are $\mathcal{H}_k = \{\mathcal{Y}_k^i, i = 1, ..., 2k+1\}$- spanned by spherical harmonics of degree k. T^k shall denote the irreducible representation, obtained by restriction of T to $\mathcal{H}_k$. The matrix coefficients of T^k are the Wigner polynomials T_{ij}^k of degree k:

$$\mathcal{Y}_k^j(g^{-1} \cdot \xi) = \sum_{i=1}^{2k+1} T_{ij}^k(g)\mathcal{Y}_k^i(\xi) \qquad T_{ij}^k(g) = \langle \mathcal{Y}_k^j(g^{-1}\cdot), \mathcal{Y}_k^i(\cdot)\rangle_{L^2(S^2)}. \tag{15}$$

By construction the projection matrix on the Fourier handside given in (8) is $\pi_{SO(2)}(k) = diag(1, 0, ..., 0)$ where number of zeros is $2k$ and $\pi_{SO(2)}(k)$ of dimension $(2k+1) \times (2k+1)$. Since matrix coefficients always have the norm $\frac{1}{d_\pi}$, where d_π is the dimension of the representation, we have

$$T_{i1}^k(g) = \sqrt{\frac{4\pi}{2k+1}}\mathcal{Y}_k^i(g \cdot \xi_0), \tag{16}$$

where $\xi_0 \in S^2$ is the base point of $SO(3)/SO(2) \sim S^2$, often chosen as north pole and its stabilizer is the factorized subgroup $SO(2)$.

The eigenvalue of Laplacian on $SO(3)$ and on S^2 corresponding to polynomials of degree k is $-k(k+1)$:

$$\Delta_{SO(3)}T_{ij}^k = -k(k+1)T_{ij}^k, \quad \Delta_{S^2}\mathcal{Y}_k^i = -k(k+1)\mathcal{Y}_k^i. \tag{17}$$

Definition 3.1. *The subgroup $\mathscr{H}$ is called massive, if* $\mathrm{rank}\hat{\mathscr{H}}(\pi) \leq 1$ *for all* $\pi \in \hat{\mathcal{G}}$. *Furthermore, an irreducible representation $\pi \in \hat{\mathcal{G}}$ is called class-1 representation with respect to the subgroup $\mathscr{H}$, if* $\mathrm{rank}\hat{\mathscr{H}}(\pi) \geq 1$.

Lemma 3.2. ([36, Chapter IX.2.6]) *$SO(n)$ is a massive subgroup of $SO(n+1)$. Furthermore, the family T^k, $k \in \mathbb{N}_0$, gives up to equivalence all class-1 representations of $SO(n+1)$ with respect to $SO(n)$.*

For the following we fix the 'north pole' ξ_0 of S^2. Then the set of zonal spherical harmonics is one-dimensional and spanned by the Gegenbauer polynomials. Further the dimension of zonal functions in $\mathcal{H}_k$ is one for all $k \geq 0$ and it is spanned by Gegenbauer polynomial of order $C_k^{\frac{1}{2}}(\xi_0 \cdot \xi)$, note that $\cos(\angle(\xi_0, \xi) = \xi_0 \cdot \xi)$. Hence, to be zonal on S^2 for a function $f(\xi)$ means to be invariant under the action of $SO(2)$, i.e. to depend only on the angle between the argument ξ and the $SO(2)$ invariant point ξ_0.

The following addition theorem holds true

Theorem 3.3 (Addition theorem)**.** *For all $\xi, \eta \in \mathbb{S}^2$ and $k \in \mathbb{N}_0$*

$$\mathcal{C}_k^{\frac{1}{2}}(\xi \cdot \eta) = \frac{4\pi}{2k+1} \sum_{i=1}^{2k+1} \mathcal{Y}_k^i(\xi)\overline{\mathcal{Y}_k^i(\eta)}. \tag{18}$$

3.2. **Radon transform on** $SO(3)$**.** The Radon transform on $SO(3)$ is defined in Definition 1.1. We also need apropriate function spaces, which adjust the smoothness of the functions. Below we define Sobolev spaces for $S^2 \times S^2$.

Definition 3.4. *The Sobolev space* $H_t(S^2 \times S^2)$, $t \in \mathbb{R}$, *is defined as the domain of the operator* $(1 - 2\Delta_{S^2 \times S^2})^{\frac{t}{2}}$ *with graph norm*

$$||f||_t = ||(1 - 2\Delta_{S^2 \times S^2})^{\frac{t}{2}} f||_{L^2(S^2 \times S^2)}, \;\; f \in L^2(S^2 \times S^2).$$

Since

$$\mathscr{R}(T^k)(x, y) = T^k(x)\pi_{SO(2)}(T^k(y))^* \tag{19}$$

we have

$$\mathscr{R}T^k_{ij}(\xi, \eta) = T^k_{i1}(\xi)\overline{T^k_{j1}(\eta)} = \frac{4\pi}{2k+1}\mathcal{Y}^i_k(\xi)\overline{\mathcal{Y}^j_k(\eta)}. \tag{20}$$

This formula shows that range of $\mathscr{R}$ belongs to kernel of the Darboux-type operator i.e.

$$\Delta_x \mathscr{R} f(x, y) = \Delta_y \mathscr{R} f(x, y), \quad f \in L_2(SO(3)). \tag{21}$$

Because $\mathscr{R}f$ is in the kernel of the Darboux-type operator we also need the following Sobolev-type function space.

Definition 3.5. *The Sobolev space* $H^{\Delta}_t(S^2 \times S^2)$, $t \in \mathbb{R}$, *is defined as the subspace of all functions* $f \in H_t(S^2 \times S^2)$ *such* $\Delta_1 f = \Delta_2 f$.

Now we define Sobolev spaces on $SO(3)$.

Definition 3.6. *The Sobolev space* $H_t(SO(3))$, $t \in \mathbb{R}$, *is defined as the domain of the operator* $(1 - 4\Delta_{SO(3)})^{\frac{t}{2}}$ *with graph norm*

$$|||f|||_t = ||(1 - 4\Delta_{SO(3)})^{\frac{t}{2}} f||_{L^2(SO(3))}, \;\; f \in L^2(SO(3)).$$

Because the operators $1 - 2\Delta_{S^2 \times S^2}$ and $1 - 4\Delta_{SO(3)}$ have positive spectrum and one can choose corresponding eigenfunctions which form an orthonormal basis. These definitions are consistent with another norm on the Sobolev space $H_t(M)$ which will be given in Definition 4.1.

Theorem 3.7. *(Range description) For any* $t \geq 0$ *the Radon transform on* $SO(3)$ *is an invertible mapping*

$$\mathscr{R} : H_t(SO(3)) \to H^{\Delta}_{t+\frac{1}{2}}(S^2 \times S^2). \tag{22}$$

Proof. It is sufficient to consider case $t = 0$. If $d_k = 2k + 1$ is the dimension of the irreducible representations and $-\lambda_k^2 = -k(k + 1)$ are the eigenvalues of the Laplacian $\Delta_{SO(3)}$ we have $d_k = \sqrt{1 + 4\lambda_k^2}$. Since $\mathscr{H} = SO(2)$ is massive in $\mathcal{G} = SO(3)$ and T^k are class-1 representations of $SO(3)$ with respect to $SO(2)$ we have $\mathrm{rank}(\pi_{\mathscr{H}}) = 1$ and $\widehat{\mathcal{G}}_1 = \widehat{\mathcal{G}}$. Now the assertion follows from (12) and (13). We

have

$$\begin{aligned}\|\mathscr{R}f\|^2_{\frac{1}{2}} &= \|(1-2\Delta_{S^2\times S^2})^{\frac{1}{4}}\mathscr{R}f\|^2_{L^2(S^2\times S^2)} = \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi \int_{\mathcal{G}} \|\sqrt{d_\pi}\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\|^2_{HS}\,\mathrm{d}y \\ &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi^2 \int_{\mathcal{G}} \mathrm{trace}\left(\widehat{f}^*(\pi)\pi(y)\pi_{\mathscr{H}}\pi^*(y)\widehat{f}(\pi)\right)\,\mathrm{d}y \\ &= \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi^2\,\mathrm{trace}\left(\widehat{f}^*(\pi)\int_{\mathcal{G}}\pi(y)\pi_{\mathscr{H}}\pi^*(y)\,\mathrm{d}y\,\widehat{f}(\pi)\right) \\ &= \sum_{\pi\in\widehat{\mathcal{G}}_1=\widehat{\mathcal{G}}} d_\pi \mathrm{rank}(\pi_{\mathscr{H}})\,\mathrm{trace}(\widehat{f}^*\widehat{f}) = \sum_{\pi\in\widehat{\mathcal{G}}} d_\pi\|\widehat{f}(\pi)\|^2_{HS} = ||f||^2_{L^2(SO(3))} = |||f|||^2_0\end{aligned}$$

□

From Theorem 3.7 we deduce the reconstruction formula for the Radon transform on $SO(3)$. This is also the result of a simple calculation involving spherical harminoncs as well as Wigner polynomials. Using the identities (20) we get the following theorem:

Theorem 3.8. *(Reconstruction formula)*
Let

$$(\mathscr{R}g)(x,y) = f(x,y) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1}\widehat{f}(k,i,j)\mathcal{Y}_k^i(x)\overline{\mathcal{Y}_k^j(y)} \in H^{\Delta}_{\frac{1}{2}}(S^2\times S^2) \tag{23}$$

be the result of a Radon transform. Then the pre-image $g \in L^2(SO(3))$ is given by

$$g = \sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1}\frac{(2k+1)}{4\pi}\widehat{f}(k,i,j)T^k_{ij} = \sum_{k=0}^{\infty}(2k+1)\,\mathrm{trace}(\widehat{g}(k)T^k) \tag{24}$$

$$\widehat{g}(k,i,j) = \frac{1}{4\pi}\widehat{f}(k,i,j). \tag{25}$$

4. Generalized variational splines on compact Riemannian manifolds

We consider a compact Riemannian manifold M without boundary. Let $\mathcal{L}$ be a differential of order two elliptic operator which is self-adjoint and negatively semi-definite in the space $L_2(M)$ constructed using a Riemannian density dx. The spectrum of such operator always contains $\lambda_0 = 0$. In order to have an invertible operator we will work with $I - \mathcal{L}$, where I is the identity operator in $L_2(M)$. It is known that for every such operator $\mathcal{L}$ the domain of the power $(-\mathcal{L})^{t/2}$, $t \in \mathbb{R}$, is the Sobolev space $H_t(M)$. There are different ways to introduce norm in Sobolev spaces. We choose the following definition.

Definition 4.1. *The Sobolev space $H_t(M), t \in \mathbb{R}$ can be introduced as the domain of the operator $(1-\mathcal{L})^{t/2}$ with the graph norm*

$$\|f\|_t = \|(1-\mathcal{L})^{t/2}f\|, f \in H_t(M).$$

Note, that such norm depends on $\mathcal{L}$. However, for every two differential of order two elliptic operators such norms are equivalent for each $t \in \mathbb{R}$.

Since the operator $(-\mathcal{L})$ is self-adjoint and positive semi-definite it has a discrete spectrum $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ...$, and one can choose corresponding eigen functions

$\varphi_0, \varphi_1, ...$ which form an orthonormal basis of $L_2(M)$. A distribution f belongs to $H_t(M), t \in \mathbb{R}$, if and only if

$$\|f\|_t = \left(\sum_{j=0}^{\infty}(1+\lambda_j)^t|c_j(f)|^2\right)^{1/2} < \infty,$$

where Fourier coefficients $c_j(f)$ of f are given by

$$c_j(f) = \langle f, \varphi_j \rangle = \int_M f\overline{\varphi_j}.$$

This L_2-inner product can be also considered as a pairing between $H_{-t}(M)$ and $H_t(M)$ and in this sense every element of $H_{-t}(M)$ can be identified with a continuous functional on $H_t(M)$.

For a given finite family of pairwise different submanifolds $\{M_\nu\}_1^N$ consider the following family of distributions

$$F_\nu(f) = \int_{M_\nu} f \tag{26}$$

which are well defined at least for functions in $H_{\varepsilon+d/2}(M),\ \varepsilon > 0$. In particular, if $M_\nu = x_\nu \in M$, then every F_ν is a Dirac measure δ_{x_ν} $\nu = 1, ..., N,\ x_\nu \in M$.

Note that distributions F_ν belong to $H_{-\varepsilon-d/2}(M)$ for any $\varepsilon > 0$.

Variational Problem. *Given a sequence of complex numbers $v = \{v_\nu\}$, $\nu = 1, 2, ..., N$, and a $t > d/2$ we consider the following variational problem:*

Find a function u from the space $H_t(M)$ which has the following properties:

1. $F_\nu(u) = v_\nu, \nu = 1, 2, ..., N, v = \{v_\nu\}$,
2. *u minimizes functional $u \to \|(1-\mathcal{L})^{t/2}u\|$.*

We show that the solution to Variational problem exist and is unique for any $t > t_0$. We need the following Independence Assumption in order to determine the Fourier coefficients of the solution.

Independence Assumption. *There are functions $\vartheta_\nu \in C^\infty(M)$ such that*

$$F_\nu(\vartheta_\mu) = \delta_{\nu\mu}, \tag{27}$$

where $\delta_{\nu\mu}$ is the Kronecker delta.

Note, that this assumption implies in particular that the functionals F_ν are linearly independent. Indeed, if we have that for certain coefficients $\gamma_1, \gamma_2, ..., \gamma_N$

$$\sum_{\nu=1}^{N}\gamma_\nu F_\nu = 0,$$

then for any $1 \le \mu \le N$

$$0 = \sum_{\nu=1}^{N}\gamma_\nu F_\nu(\vartheta_\mu) = \gamma_\mu.$$

The families of distributions that satisfy our condition include

a) Finite families of δ functionals and their derivatives.

b) Sets of integrals over submanifolds from a finite family of submanifolds of any codimension.

The solution to the Variational Problem will be called a spline and will be denoted as $s_t(v)$. The set of all solutions for a fixed set of distributions $F = \{F_\nu\}$ and a fixed t will be denoted as $S(F, t)$.

Definition 4.2. *Given a function $f \in H_t(M)$ we will say that the unique spline s from $S(F, t)$ interpolates f if*

$$F_\nu(f) = F_\nu(s).$$

Such spline will be denoted as $s_t(f)$.

From the point of view of the classical theory of variational splines it would be more natural to consider minimization of the functional

$$u \to \|\mathcal{L}^{t/2} u\|.$$

However, in the case of a general compact manifolds it is easer to work with the operator $1-\mathcal{L}$ since this operator is invertible. The following existence and uniqueness theorem was proved in [34].

Theorem 4.3. *The Variational Problem has a unique solution for any sequence of values $(v_1, v_2, ...v_N)$.*

Proof. Consider the set

$$\bigcap_\nu Ker\, F_\nu = V_t^0(F) \subset H_t(M), t > d/2,$$

of all functions in $H_t(M)$ such that for every $1 \leq \nu \leq N, F_\nu(f) = 0$.

Given a sequence of complex numbers $(v_1, v_2, ..., v_N)$ the linear manifold

$$V_t(F, v_1, ...v_N), t > d/2$$

of all functions f in $H_t(M)$ such that $F_\nu(f) = v_\nu, \nu = 1, ..., N$, is a shift of the closed subspace $V_t^0(F)$, i.e.

$$V_t(F, v_1, ..., v_N) = V_t^0(F) + g,$$

where g is any function from $H_t(M)$ such that $F_\nu(g) = v_\nu, \nu = 1, 2, ..., N$.

Consider the orthogonal projection g_0 of $g \in H_t(M)$ onto the space $V_t^0(F)$ with respect to the inner product in $H_t(M)$:

$$\langle f_1, f_2 \rangle_{H_t(M)} = \left\langle (1-\mathcal{L})^{t/2} f_1, (1-\mathcal{L})^{t/2} f_2 \right\rangle_{L_2(M)} =$$

$$\int_M (1-\mathcal{L})^{t/2} f_1 \overline{(1-\mathcal{L})^{t/2} f_2}.$$

Note, that $s_t(v) = g - g_0 \in V_t(F, v_1, ..., v_N)$ is the unique solution of the Variational Problem. Indeed, to show that $s_t(v)$ minimizes the functional

$$u \to \|(1-\mathcal{L})^{t/2} u\|$$

on the set $V_t(F, v_1, ..., v_N)$ we note that any function in $V_t(F, v_1, ..., v_N)$ can be written in the form $s_t(v) + h$, where $h \in V_t^0(F)$. For such a function we have

$$\|(1-\mathcal{L})^{t/2}(s_t(v) + h)\|^2 =$$

$$\|(1-\mathcal{L})^{t/2}s_t(v)\|^2+2\left\langle s_t(v),h\right\rangle_{H_t(M)}+\|(1-\mathcal{L})^{t/2}h\|^2.$$

Since $s_t(v)=g-g_0$ is orthogonal to $V_t^0(F)$ we obtain

$$\|(1-\mathcal{L})^{t/2}(s_t(v)+\sigma h)\|^2=\|(1-\mathcal{L})^{t/2}s_t(v)\|^2+|\sigma|^2\|(1-\mathcal{L})^{t/2}h\|^2,\ \ h\in V_t^0(F),$$

that shows that the function $s_t(v)$ is the minimizer. □

The following criterion follows from the previous theorem [34].

Theorem 4.4. *A function $u\in H_t(M)$ is a solution of the Variational Problem if and only if it is orthogonal to the subspace $V_t^0(F)$ and $F_\nu(u)=v_\nu,\nu=1,2,....$*

As a consequence we obtain that the set of all solutions of the Variational Problem is linear. In particular, every spline $s_t(v)\in S(F,t)$ has the following representation through its values $F_\nu(s_t(v))=v_\nu,\nu=1,...,N$, on X:

$$s_t(v)=\sum_{\nu=1}^{N}v_\nu l^\nu, \tag{28}$$

where $F_\nu(s_t(v))=v_\nu$, and $l^\nu\in S(F,t),\nu=1,2,...,N$, is so called Lagrangian spline that defined by conditions $F_\mu(l^\nu)=\delta_{\nu\mu},\mu=1,2,...,N$.

The next Theorem gives the characteristic property of splines.

Theorem 4.5. *A function $s_t(v)\in H_t(M),t>d/2$ is a solution of the Variational Problem if and only if it satisfies the following equation in the sense of distributions*

$$(1-\mathcal{L})^t s_t(v)=\sum_{\nu=1}^{N}\alpha_\nu(s_t(v))\overline{F_\nu}. \tag{29}$$

In other words, for any smooth ψ

$$\left\langle(1-\mathcal{L})^t s_t(v),\psi\right\rangle_{L_2(M)}=\int_M(1-\mathcal{L})^t s_t(v)\overline{\psi}=\sum_{\nu=1}^{N}\alpha_\nu(s_t(v))\overline{F_\nu(\psi)}.$$

Proof. We already know that every solution of the Variational Problem is orthogonal to $V_t^0(F)$ in the Hilbert space $H_t(M)$ i.e. for any $h\in V_t^0(F)$

$$0=\langle s_t(v),h\rangle_{H_t(M)}=\int_M(1-\mathcal{L})^{t/2}s_t(v)\overline{(1-\mathcal{L})^{t/2}h}. \tag{30}$$

Note, that since we consider a **finite** family of pairwise different manifolds M_ν our **Independence Assumption** (27) is satisfied: there are functions $\vartheta_\nu\in C^\infty(M)$ such that

$$F_\nu(\vartheta_\mu)=\delta_{\nu\mu}, \tag{31}$$

where $\delta_{\nu\mu}$ is the Kronecker delta. Thus, for any $\psi\in C_0^\infty(M)$ the function

$$\psi-\sum_{\nu=1}^{N}F_\nu(\psi)\vartheta_\nu$$

belongs to $V_t^0(F)$ and because of (30)

$$0=\left\langle s_t(v),\psi-\sum_{\nu=1}^{N}F_\nu(\psi)\vartheta_\nu\right\rangle_{H_t(M)}=\int_M(1-\mathcal{L})^{t/2}s_t(v)\overline{(1-\mathcal{L})^{t/2}(\psi-\sum_{\nu=1}^{N}F_\nu(\psi)\vartheta_\nu)}$$
$$=\int_M(1-\mathcal{L})^{t}s_t(v)\left(\overline{\psi-\sum_{\nu=1}^{N}F_\nu(\psi)\vartheta_\nu}\right).$$

In other words,

$$\int_M\left[(1-\mathcal{L})^{t}s_t(v)\right]\overline{\psi}=\sum_{\nu=1}^{N}\overline{F_\nu(\psi)}\int_M(1-\mathcal{L})^{t}s_t(v)\overline{\vartheta_\nu}.$$

If we set

$$\alpha_\nu(s_t(v),\vartheta)=\int_M\left[(1-\mathcal{L})^{t}s_t(v)\right]\overline{\vartheta_\nu},$$

we obtain that $(1-\mathcal{L})^{t}s_t(v)$ is a distribution of the form

$$(1-\mathcal{L})^{t}s_t(v)=\sum_{\nu=1}^{N}\alpha_\nu(s_t(v),\vartheta)\overline{F_\nu},$$

where

$$\overline{F_\nu}(\psi)=\overline{F_\nu(\psi)}.$$

So every solution of the variational problem is a solution of (29).

Note, that if $\zeta=\{\zeta_\nu\}$ is another $C_0^\infty(M)-$family for which $F_\nu(\zeta_\mu)=\delta_{\nu\mu}$, then we have the identity

$$\sum_{\nu=1}^{N}(\alpha_\nu(s_t(v),\vartheta)-\alpha_\nu(s_t(v),\zeta))F_\nu=0.$$

Since distributions F_ν are linearly independent it implies that

$$\alpha_\nu(s_t(v),\vartheta)-\alpha_\nu(s_t(v),\zeta)=0.$$

In other words, coefficients $\alpha_\nu(s_t(v),\vartheta)=\alpha_\nu(s_t(v),\zeta)=\alpha_\nu(s_t(v))$ are independent of the choice of the family of functions ϑ.

Conversely, if u is a solution of (29) then since F_ν belongs to the space $H_{-t_0}(M)$, and $t>t_0\geq 0$, the Regularity Theorem for elliptic operator $(1-\mathcal{L})^t$ implies that $u\in H_{-t_0+2t}(M)\subset H_t(M)$ and for any $h\in V_t^0(F)$

$$\langle u,h\rangle_{H_t(M)}=\left\langle(1-\mathcal{L})^{t/2}u,(1-\mathcal{L})^{t/2}h\right\rangle=\left\langle(1-\mathcal{L})^{t}u,h\right\rangle$$
$$=\sum_{\nu=1}^{N}\alpha_\nu(u)F_\nu(h)=0,$$

that shows that u is a the solution of the Variational Problem. □

The formula (28) represents spline through its values and Lagrangian splines. To obtain another representation of splines we will need what we call generalized fundamental solutions or generalized Green's functions E_ν^t, which are solutions of the following distributional equations

$$(1-\mathcal{L})^{t}E_\nu^t=\overline{F_\nu}. \tag{32}$$

To find E^t_ν we note that in the sense of distributions

$$\overline{F_\nu}=\sum_{j=0}^{\infty}\overline{F_\nu(\varphi_j)}\varphi_j \tag{33}$$

which shows that

$$E^t_\nu=\sum_{j=0}^{\infty}(1+\lambda_j)^{-t}\overline{F_\nu(\varphi_j)}\varphi_j. \tag{34}$$

The following important fact holds.

Theorem 4.6. *Every spline $s_t(v)$ is a linear combination of the generalized fundamental solutions*

$$s_t(v)=\sum_{\nu=1}^{N}\alpha_\nu(s_t(v))E^t_\nu \tag{35}$$

where scalar coefficients $\alpha_\nu(s_t(v))$ are the same as in Theorem 4.5.

Proof. According to Theorem 4.5 every spline $s_t(v)$ is a solution of

$$(1-\mathcal{L})^t s_t(v)=\sum_{\nu=1}^{N}\alpha_\nu(s_t(v))\overline{F_\nu}.$$

Thus, we obtain the equality

$$(1-\mathcal{L})^t s_t(v)=\sum_{\nu}\alpha_\nu(s_t(v))\sum_{j}\overline{F_\nu(\varphi_j)}\varphi_j,$$

that implies the desired representation. □

Note that so far we have used just the assumption that $t>d/2$. To get more information about $s_t(v)$ we will need a stronger assumption that $t>d$.

Theorems 4.3 and 4.6 imply our main result concerning variational splines (see [34]).

Theorem 4.7. *If $t>d$, then for any given sequence of scalars $v=\{v_\nu\},\nu=1,2,...N$, the following statements are equivalent:*

1) $s_t(v)$ is the solution to the Variational Problem;

2) $s_t(v)$ satisfies the equation (29) in the sense of distributions

$$(1-\mathcal{L})^t s_t(v)=\sum_{\nu=1}^{N}\alpha_\nu(s_t(v))\overline{F_\nu},t>d, \tag{36}$$

where $\alpha_1(s_t(v)),...,\alpha_N(s_t(v))$ form a solution of the $N\times N$ system

$$\sum_{\nu=1}^{N}\beta_{\nu\mu}\alpha_\nu(s_t(v))=v_\mu,\mu=1,...,N, \tag{37}$$

and

$$\beta_{\nu\mu}=\sum_{j=0}^{\infty}(1+\lambda_j)^{-t}\overline{F_\nu(\varphi_j)}F_\mu(\varphi_j),\quad \mathcal{L}\varphi_j=-\lambda\varphi_j; \tag{38}$$

3) the Fourier series of $s_t(v)$ has the following form

$$s_t(v) = \sum_{j=0}^{\infty} c_j(s_t(v))\varphi_j, \tag{39}$$

where

$$c_j(s_t(v)) = \langle s_t(v), \varphi_j \rangle = (1+\lambda_j)^{-t} \sum_{\nu=1}^{N} \alpha_\nu(s_t(v)) \overline{F_\nu(\varphi_j)}.$$

Remark 4.8. *It is important to note that the system (37) is always solvable according to our uniqueness and existence result for the Variational Problem.*

Remark 4.9. *It is also necessary to note that the series (38) is absolutely convergent if $t > d$. Indeed, since functionals F_ν are continuous on the Sobolev space $H_{d/2+\varepsilon}(M)$ we obtain that for any normalized eigen function φ_j which corresponds to the eigen value λ_j the following inequality holds true*

$$|F_\nu(\varphi_j)| \leq C(M, F)\|(1-\mathcal{L})^{d/4}\varphi_j\| \leq C(M, F)(1+\lambda_j)^{d/4}, \quad F = \{F_\nu\}.$$

So

$$|\overline{F_\nu(\varphi_j)}F_\mu(\varphi_j)| \leq C(M, F)(1+\lambda_j)^{d/2},$$

and

$$|(1+\lambda_j)^{-t}\overline{F_\nu(\varphi_j)}F_\mu(\varphi_j)| \leq C(M, F)(1+\lambda_j)^{(t_0-t)}.$$

It is known [26] *that the series*

$$\sum_j \lambda_j^{-\tau},$$

which defines the ζ−function of an elliptic second order operator, converges if $\tau > d/2$. This implies absolute convergence of (38) in the case $t > d$.

One can show that splines provide an optimal approximations to sufficiently smooth functions. Namely let $Q(F, f, t, K)$ be the set of all functions g in $H_t(M)$ such that

(1) $F_\nu(g) = F_\nu(f), \nu = 1, 2, ..., N$,
(2) $\|g\|_t \leq K$, for a real $K \geq \|s_t(f)\|_t$.

The set $Q(F, f, t, K)$ is convex, bounded and closed.

The following theorem (see [34]) shows that splines provide an optimal approximations to functions in $Q(F, f, t, K)$.

Theorem 4.10. *The spline $s_t(f)$ is the symmetry center of $Q(F, f, t, K)$. This means that for any $g \in Q(F, f, t, K)$*

$$\|s_t(f) - g\|_t \leq \frac{1}{2} diam\, Q(F, f, t, K). \tag{40}$$

5. Approximate inversion of the group Radon transform using generalized variational interpolating splines

We return to the Radon transform on a compact Lie group $\mathcal{G}$. In order to apply the general scheme developed in the previous section we select a finite number of pairs $(x_\nu, y_\nu) \in \mathcal{G} \times \mathcal{G}, \nu = 1, ..., N$, and introduce submanifolds $\mathcal{M}_\nu = x_\nu \mathscr{H} y_\nu^{-1} \subset$

$\mathcal{G}$. Note, that for any $\nu = 1, ..., N$ the dimension $dim\, \mathcal{M}_\nu = d_\nu$ equals $dim\, \mathscr{H}$. Next, for the set of scalars

$$v_\nu = \mathscr{R}f(x_\nu, y_\nu) = \int_{\mathscr{H}} f(x_\nu h y_\nu^{-1})\, \mathrm{d}h = \int_{\mathcal{M}_\nu} f(h)\mathrm{d}h \tag{41}$$

we consider the following variational problem: for a $t > \frac{1}{2} dim\, \mathcal{G}$ find a function u in the space $H_t(\mathcal{G})$ which satisfies (41) and minimizes the functional

$$u \to \|(1 - \Delta_{\mathcal{G}})^{t/2} u\|.$$

According to Theorem 4.7 the solution $s_t = s_t(v),\ v = \{v_\nu\}$, to this problem is given by the formula

$$s_t = \sum_{\pi \in \widehat{\mathcal{G}}} \sum_{i,j=0}^{d_\pi} c_{ij}^\pi \pi_{ij}, \tag{42}$$

where

$$c_{ij}^\pi = c_{ij}^\pi(s_t(v)) = (1 + \lambda_\pi^2)^{-t} \sum_{\nu=1}^{N} \alpha_\nu \pi_{ij}(x_\nu, y_\nu),\ \ \alpha_\nu = \alpha_\nu(s_t(v)). \tag{43}$$

Coefficients $\alpha_1, ..., \alpha_N$ are solutions of the following system

$$\beta_{1\mu}\alpha_1 + ... + \beta_{N\mu}\alpha_N = v_\mu,\ \ \mu = 1, ..., N. \tag{44}$$

To determine matrix β with entries $\beta_{\nu\mu},\ \nu, \mu = 1, ..., N$, one only needs to find the data $\mathscr{R}\pi(g_\nu)$ and to compute quantities $\sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu, y_\nu))} \mathscr{R}(\pi_{ij}(x_\mu, y_\mu))$. The entries $\beta_{\nu\mu}$ are given by the formulas

$$\beta_{\mu\nu} = \sum_{\pi \in \widehat{\mathcal{G}}} (1 + \lambda_\pi^2)^{-t} \sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu, y_\nu))} \mathscr{R}(\pi_{ij}(x_\mu, y_\mu)). \tag{45}$$

The appearing formulae for special applications will end up in well known calculations of special function, since these arise naturally from representation theory. Implementing a fast algorithm for the solution of (37), that is a standard problem, will give the solution of our variational problem for the Radon transform.

Here we have

(46)
$$\sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu, y_\nu))} \mathscr{R}(\pi_{ij}(x_\mu, y_\mu)) = \sum_{i,j=1}^{d_\pi} \int_{\mathscr{H}} \overline{\pi_{ij}(x_\nu h y_\nu^{-1})\, \mathrm{d}h} \int_{\mathscr{H}} \pi_{ij}(x_\mu h y_\mu^{-1})\, \mathrm{d}h$$

$$= \sum_{i,j=1}^{d_\pi} \int_{\mathscr{H}} \int_{\mathscr{H}} \pi_{ji}(y_\nu h x_\nu^{-1})\, \mathrm{d}h \pi_{ij}(y_\mu h x_\mu^{-1})\, \mathrm{d}h \tag{47}$$

$$= \mathrm{trace}(\pi_{\mathscr{H}} \pi(y_\nu) \pi_{\mathscr{H}} \pi(x_\nu^{-1}) \pi_{\mathscr{H}} \pi(x_\mu) \pi_{\mathscr{H}} \pi(y_\mu^{-1})), \tag{48}$$

hence we obtain function that is zonal in every component. A special case of that is the Addition theorem 3.3 for spherical harmonics.

In the case where $\mathscr{H}$ is a massive subgroup of $\mathcal{G}$ we find

$$\sum_{i,j=1}^{d_\pi} \overline{\mathscr{R}(\pi_{ij}(x_\nu, y_\nu))}\mathscr{R}(\pi_{ij}(x_\mu, y_\mu)) = \pi_{11}(y_\nu)\pi_{11}(x_\nu^{-1})\pi_{11}(x_\mu)\pi_{11}(y_\mu^{-1}) \tag{49}$$

$$= \pi_{11}(y_\nu x_\nu^{-1} x_\mu y_\mu^{-1}), \tag{50}$$

and hence

$$\beta_{\mu\nu} = \sum_{\pi \in \widehat{\mathcal{G}}_1} (1+\lambda_\pi^2)^{-t} \pi_{11}(y_\nu x_\nu^{-1} x_\mu y_\mu^{-1}) \tag{51}$$

where $\widehat{\mathcal{G}}_1$ denotes the set of irreducible representations with rank $\pi_{\mathscr{H}} = 1$.

Since $SO(2)$ is a massive subgroup in $SO(3)$ the last formula can be used in the case of the Radon transform on $SO(3)$.

Let $\{(x_1, y_1), ..., (x_N, y_N)\}$ be a set of pairs of points from $\mathcal{G}$. In what follows we have to assume that our **Independence Assumption** (27) holds. It takes now the following form: there are smooth functions $\phi_1, ..., \phi_N$ on $\mathcal{G}$ with

$$\mathscr{R}\phi_\mu(x_\nu, y_\nu) = \delta_{\nu\mu}.$$

But it is obvious that for this condition to satisfy it is enough to assume that submanifolds $\mathcal{M}_\nu = x_\nu \mathscr{H} y_\nu^{-1} \subset \mathcal{G}$ are pairwise different (not necessarily disjoint).

Let f be in a continuous function on $\mathcal{G}$, $\quad t > \frac{1}{2} dim \mathcal{G}, \quad v = \{v_\nu\}_1^N$ where

$$v_\nu = \int_{\mathcal{M}_\nu} f.$$

According to Definition 4.2 we use notation $s_t(f) = s_t(v)$ for a function in $H_t(\mathcal{G})$ such that for $\mathcal{M}_\nu = x_\nu \mathscr{H} y_\nu^{-1}$ one has

$$(\mathscr{R}f)(x_\nu, y_\nu) = \int_{\mathcal{M}_\nu} f = \int_{\mathcal{M}_\nu} s_t(f) = (\mathscr{R}s_t(f))(x_\nu, y_\nu), \tag{52}$$

and

$$\|s_t(f)\|_{H_t(\mathcal{G})} \to \min. \tag{53}$$

Theorem 5.1. *Let $\{(x_1, y_1), ..., (x_N, y_N)\}$ be a set of pairs of points from $\mathcal{G}$, such that submanifolds $\mathcal{M}_\nu = x_\nu \mathscr{H} y_\nu^{-1} \subset \mathcal{G}$, $\nu = 1, ..., N$, are pairwise different.*

Given a continuous function f on $\mathcal{G}$ and a $t > \frac{1}{2} dim\, \mathcal{G}$ the solution of (52)-(53) is given by the formula (42). The Fourier coefficients $c_k(s_t(f))$ of the solution are given by their matrix entries (43), where $\alpha(s_t(f)) = (\alpha_\nu(s_t(f)))_1^N \in \mathbb{R}^N$ is the solution of (44) with $\beta \in \mathbb{R}^{N\times N}$ given by (45).

The function $s_t(f) \in H_t(\mathcal{G})$ has the following properties:

(1) *$s_t(f)$ has the prescribed set of measurements*

$$\int_{\mathcal{M}_\nu} f = \int_{\mathcal{M}_\nu} s_t(f);$$

(2) *it minimizes the functional*

$$u \to \|(1-\Delta_{\mathcal{G}})^{t/2} u\|;$$

(3) *the solution (42) is optimal in the sense that for every sufficiently large $K > 0$ it is the symmetry center of the convex bounded closed set of all functions g in $H_t(\mathcal{G})$ with $\|g\|_t \leq K$ which have the same set of measurements*

$$\int_{\mathcal{M}_\nu} f = \int_{\mathcal{M}_\nu} g.$$

Let us turn to the case of $SO(3)$. Since we have (20)

$$\mathscr{R}T^k_{ij}(\xi,\eta) = T^k_{i1}(\xi)\overline{T^k_{j1}(\eta)} = \frac{4\pi}{2k+1}\mathcal{Y}^i_k(\xi)\overline{\mathcal{Y}^j_k(\eta)} \tag{54}$$

it implies

$$\beta_{\nu\mu} = \sum_{k=0}^{\infty}(1+k(k+1))^{-t}\left(\frac{4\pi}{2k+1}\right)^2 \sum_{i,j=1}^{2k+1}\overline{\mathcal{Y}^i_k(x_\nu)}\mathcal{Y}^j_k(y_\nu)\mathcal{Y}^i_k(x_\mu)\overline{\mathcal{Y}^j_k(y_\mu)} \tag{55}$$

$$= \sum_{k=0}^{\infty}(1+k(k+1))^{-t}C_k^{\frac{1}{2}}(x_\nu\cdot y_\nu)C_k^{\frac{1}{2}}(x_\mu\cdot y_\mu), \tag{56}$$

where we made use of Addition theorem 3.3.

At this point one uses standard methods to solve the last system, then to obtain coefficients α_ν in (44), coefficients c^π_{ij} in (43) and obtain the solution (42).

Consider the constrained variational problem (52)-(53) for $\mathcal{G} = SO(3)$, $\mathscr{H} = SO(2)$. Below we formulate a theorem which summarizes our results for $SO(3)$.

Theorem 5.2. *Let $\{(x_1,y_1),...,(x_N,y_N)\}$ be a set of pairs of points from $SO(3)$, such that submanifolds $\mathcal{M}_\nu = x_\nu SO(2)y_\nu^{-1} \subset SO(3)$, $\nu = 1,...,N$, are pairwise different.*

For a continuous function f on $\mathcal{G}$, $t > 3/2$, and a vector (of measurements) $v = (v_\nu)_1^N$ where

$$v_\nu = \int_{\mathcal{M}_\nu} f,$$

the solution of (52)-(53) is given by

$$s_t(f) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{2k+1} c^k_{ij}(s_t(f))T^k_{ij} = \sum_{k=0}^{\infty} trace\left(c_k(s_t(f))T^k\right), \tag{57}$$

where T^k_{ij} are the Wigner polynomials. The Fourier coefficients $c_k(s_t(f))$ of the solution are given by their matrix entries

$$c^k_{ij}(s_t(f)) = \frac{4\pi}{(2k+1)(1+k(k+1))^t}\sum_{\nu=1}^{N}\alpha_\nu(s_t(f)\mathcal{Y}^i_k(x_\nu)\overline{\mathcal{Y}^j_k(y_\nu)}, \tag{58}$$

where $\alpha(s_t(f)) = (\alpha_\nu(s_t(f)))_1^N \in \mathbb{R}^N$ is the solution of

$$\beta\alpha(s_t(f)) = f, \tag{59}$$

with $\beta \in \mathbb{R}^{N\times N}$ given by

$$\beta_{\nu\mu} = \sum_{k=0}^{\infty}(1+k(k+1))^{-t}C_k^{\frac{1}{2}}(x_\nu\cdot y_\nu)C_k^{\frac{1}{2}}(x_\mu\cdot y_\mu), \tag{60}$$

where $C_k^{\frac{1}{2}}$ are the Gegenbauer polynomials.

The function $s_t(f) \in H_t(SO(3))$ has the following properties:

(1) *$s_t(f)$ has the prescribed set of measurements $v = (v_\nu)_1^N$ at points $((x_\nu, y_\nu))_1^N$;*
(2) *it minimizes the functional*
$$u \to \|(1-\Delta_{SO(3)})^{t/2}u\|;$$
(3) *the solution (57) is optimal in the sense that for every sufficiently large $K > 0$ it is the symmetry center of the convex bounded closed set of all functions g in $H_t(SO(3))$ with $\|g\|_t \leq K$ which have the same set of measurements $v = (v_\nu)_1^N$ at points $((x_\nu, y_\nu))_1^N$.*

6. Approximation by splines and a sampling theorem for Radon transform of bandlimited functions on $SO(3)$

The Theorem 4.10 already shows that variational splines provide an optimal approximation tool. However, much stronger approximation theorems by splines hold in the case when the set of distributions $F = \{F_\nu\}_1^N$ is a set of delta functions on certain set of points $\{x_\nu\}_1^N$ of M (see [34]).

Definition 6.1. *We will say that a finite set of points $X_\rho = (\xi_1, ..., \xi_N)$, $\rho > 0$, is a ρ-lattice, if*
1) The balls $B(\xi_\nu, \rho/2)$ are disjoint.
2) The balls $B(\xi_\nu, \rho)$ form a cover of M of a **fixed** *multiplicity R_M.*

The fact that the balls $B(\xi_\nu, \rho)$ form a cover of M of a **fixed** multiplicity R_M means that every point of M is covered by not more than R_M balls frm this fanily.

The fact that for any manifold of bounded geometry there exist two constants R_M and ρ_0 such that for any $\rho < \rho_0$ one can construct a ρ-lattice was proved in [30], [32], [33].

We will need the following result for functions with "many" zeros from [30], [32], [33], [34].

Lemma 6.2. *There exist constants $C(M) > 0, \rho(M) > 0$ such that for any $\rho < \rho(M)$, any ρ-lattice $X_\rho = \{\xi_\nu\}$ and for any $f \in H_{2d}(M)$ such that $f(\xi_\nu) = 0$ for all $\xi_\nu \in X_\rho$, the following inequality holds true*
$$\|f\| \leq C(M)\rho^{2d}\|(1-\mathcal{L})^d f\|, d = \dim M.$$

The next theorem extends the last estimate to higher Sobolev norms [33], [34].

Theorem 6.3. *There exist constants $C(M) > 0, \rho(M) > 0$, such that for any $0 < \rho < \rho(M)$, any ρ-lattice $X_\rho = \{\xi_\nu\}$, any smooth f which is zero on X_ρ and any $t \geq 0$*
$$\|(1-\mathcal{L})^t f\| \leq \left(C(M)\rho^{2d}\right)^{2^m} \|(1-\mathcal{L})^{2^m d+t} f\|, t \geq 0$$
for all $m = 0, 1, ...$

It is important to stress, that the constant in the last inequality is growing exponentially when order of smoothness is increasing.

Now we can formulate and prove our Approximation Theorem [33], [34].

Theorem 6.4. *There exist constants $C(M) > 0$, $\rho(M) > 0$ such that for any $0 < \rho < \rho(M)$, any ρ-lattice X_ρ, any smooth function f and any $t \geq 0$ the following inequality holds true*
$$\|(1-\mathcal{L})^t(s_{2^m d+t}(f) - f)\| \leq \left(C(M)\rho^2\right)^{2^m d} \|(1-\mathcal{L})^{2^m+t} f\|,$$

for any $m = 0, 1, ...$

Moreover, if $t > d/2 + k$ *then there exists a* $C(M,t)$ *such that*

$$\|(s_{2^m d+t}(f)(x) - f(x))\|_{C^k(M)} \le \left(C(M,t)\rho^2\right)^{2^m d} \|(1-\mathcal{L})^{2^m d+t} f\|, m = 0, 1, \dots .$$

As it was mentioned, using standard methods of interpolation theory and the fact that Besov spaces on manifolds are interpolation spaces (real interpolation method) between two Sobolev spaces the above inequalities can be extended to Besov norms.

For a compact Lie group $SO(3)$ and its closed subgroup $SO(2)$ we consider $S^2 \times S^2$. According to Theorem 3.7 if $f \in H_t(SO(3))$ then $\mathscr{R}f \in H^{\Delta}_{t+1/2}(S^2 \times S^2)$. Also, integral of f over the manifold $x_\nu SO(2) y_\nu^{-1}$ is the value of $\mathscr{R}f$ at (x_ν, y_ν). One can apply Theorem 4.7 to the manifold $S^2 \times S^2$ and the set of distributions $F = \{F_{\nu,\mu}\}$ where

$$F_{\nu,\mu}(g) = g(x_\nu, y_\mu), \quad g \in C(S^2 \times S^2), \ (x_\nu, y_\mu) \in S^2 \times S^2$$

to construct corresponding interpolant $s_\tau(\mathscr{R}f)$ of $\mathscr{R}f$. Since $dim\ (S^2 \times S^2) = 4$ for the interpolant $s_{2^{m+2}+t}(\mathscr{R}f) \in H_{2^{m+2}+t}(S^2 \times S^2)$ Approximation Theorem gives the estimate

$$\|(1 - 2\Delta_{S^2 \times S^2})^t (s_{2^{m+2}+t}(\mathscr{R}f) - \mathscr{R}f)\|_{L_2(S^2\times S^2)} \le$$

$$\left(C\rho^2\right)^{2^{m+2}} \|(1 - 2\Delta_{S^2\times S^2})^{2^{m+2}+t} \mathscr{R}f\|_{L_2(S^2\times S^2)} =$$

$$\left(C\rho^2\right)^{2^{m+2}} \|\mathscr{R}f\|_{H_{2^{m+2}+t}(S^2\times S^2)}, \tag{61}$$

for any $m = 0, 1, ...$

Let $\widehat{s}_{2^{m+2}+t}(\mathscr{R}f)$ be orthogonal progection of $s_{2^{m+2}+t}(\mathscr{R}f)$ onto subspace

$$Range\mathscr{R} = H^{\Delta}_{1/2}(S^2 \times S^2).$$

According to (20) it means that $\widehat{s}_{2^{m+2}+t}(\mathscr{R}f)$ has a representation of the form

$$\widehat{s}_{2^{m+2}+t}(\mathscr{R}f)(\xi,\eta) = \sum_k \sum_{ij} c^k_{ij}(\mathscr{R}f; m, t) \mathcal{Y}^i_k(\xi) \overline{\mathcal{Y}^j_k(\eta)}, \quad (\xi,\eta) \in S^2 \times S^2.$$

Applying (20) one more time we obtain that the following function defined on $SO(3)$

$$S_{2^{m+2}+t}(f)(\xi) = \mathscr{R}^{-1}\left(\widehat{s}_{2^{m+2}+t}(\mathscr{R}f)\right)(\xi)$$

has a representation

$$S_{2^{m+2}+t}(f)(\xi) = \sum_k \sum_{ij} \frac{2k+1}{4\pi} c^k_{ij}(\mathscr{R}f; m, t) T^k_{ij}(\xi),$$

where T^k_{ij} are Wigner functions.

Let us stress that functions $S_\tau(f)$ do not interpolate f in any sense. However, the following approximation results hold.

Theorem 6.5. *For* $t > 1/2$ *for sufficiently smooth functions one has the estimate*

$$\|(1 - 4\Delta_{SO(3)})^{t-1/4} (S_{2^{m+2}+t}(f) - f)\|_{L_2(SO(3))} \le$$

$$\left(C\rho^2\right)^{2^{m+2}} \|\mathscr{R}f\|_{H_{2^{m+2}+t}(S^2\times S^2)} \tag{62}$$

for any $m = 0, 1, ...$

In particular, if a natural k satisfies the inequality $t > k + 7/4$, then

$$\|S_{2^{m+2}+t}(f) - f\|_{C^k(SO(3))} \leq \left(C\rho^2\right)^{2^{m+2}} \|\mathscr{R}f\|_{H_{2^{m+2}+t}(S^2\times S^2)} \tag{63}$$

for any $m = 0, 1, ...$

The proof of the first inequality follows from (59) and the fact that $\mathscr{R}^{-1}$ is continuous from $H_t^{\Delta}(S^2 \times S^2)$ to Sobolev $H_{t-1/2}(SO(3))$. The second inequality follows from the Sobolev Embedding Theorem.

For an $\omega > 0$ let us consider the space $\mathbf{E}_\omega(SO(3))$ of ω-bandlimited functions on $SO(3)$ i.e. the span of all Wigner functions T^k_{ij} with $k(k+1) \leq \omega$. As the formulas (17) and (20) show the Radon transform of such function is ω-bandlimited on $S^2 \times S^2$ in the sense its Fourier expansion involves only functions $\mathcal{Y}^k_i\overline{\mathcal{Y}^k_j}$ which are eigenfunctions of $\Delta_{S^2\times S^2}$ with eigenvalue $-k(k+1)$. Under our assumption about k the following Bernstein-type inequality holds for any function in the span of $\mathcal{Y}^k_i\overline{\mathcal{Y}^k_j}$

$$\|(1 - 2\Delta_{S^2\times S^2})^\tau \mathscr{R}f\|_{L^2(S^2\times S^2)} \leq (1 + 2\omega)^\tau \|\mathscr{R}f\|_{L^2(S^2\times S^2)}$$

Note that if $f \in \mathbf{E}_\omega(SO(3))$ then the following functions are also bandlimited

$$\widehat{s}_{2^{m+2}+t}(\mathscr{R}f)(\xi, \eta) = \sum_{k\leq\omega}\sum_{ij} c^k_{ij}(\mathscr{R}f; m, t)\mathcal{Y}^i_k(\xi)\overline{\mathcal{Y}^j_k(\eta)}, \quad (\xi, \eta) \in S^2 \times S^2,$$

$$S_{2^{m+2}+t}(f)(\xi) = \sum_{k\leq\omega}\sum_{ij} \frac{2k+1}{4\pi} c^k_{ij}(\mathscr{R}f; m, t)T^k_{ij}(\xi),$$

Using our definition (3.4) of the norm in the space $H_t(S^2 \times S^2)$ we obtain the following refinement of Theorem 6.5

Theorem 6.6. *(Sampling Theorem For Radon Transform).*
For $f \in \mathbf{E}_\omega(SO(3))$ and $t > 1/4$ one has the estimate

$$\|(1 - 4\Delta_{SO(3)})^{t-1/4}(S_{2^{m+2}+t}(f) - f)\|_{L_2(SO(3))} \leq$$

$$\left(C\rho^2(1 + 2w)\right)^{2^{m+2}} (1 + 2w)^t \|\mathscr{R}f\| \tag{64}$$

for any $m = 0, 1, ...$

In particular, if a natural k satisfies the inequality $t > k + 7/4$, then

$$\|S_{2^{m+2}+t}(f) - f\|_{C^k(SO(3))} \leq \left(C\rho^2(1 + 2w)\right)^{2^{m+2}} (1 + 2w)^t \|\mathscr{R}f\| \tag{65}$$

for any $m = 0, 1, ...$

Let us remind that the function $S_{2^{m+2}+t}(f)$ was constructed by using only the values of the Radon transform $\mathscr{R}f$ on a lattice of points on $S^2 \times S^2$. This Sampling Theorem shows that if

$$\rho < \frac{1}{\sqrt{C(1 + 2\omega)}}$$

than one can have a complete reconstruction of ω-bandlimited f when m goes to infinity by using only the values of its Radon transform $\mathscr{R}f$ on any fixed ρ-lattice of $S^2 \times S^2$.

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